Exercise 1H: Solutions

1 a

$$\begin{aligned} ax + n &= m \\ ax &= m - n \\ x &= \frac{m-n}{a} \end{aligned}$$

b

$$\begin{aligned} ax + b &= bx \\ ax - bx &= -b \\ x(a-b) &= -b \\ x &= \frac{-b}{a-b} \end{aligned}$$

This answer is correct, but to avoid a negative sign, multiply numerator and denominator by -1.

$$\begin{aligned} x &= \frac{-b}{a-b} \times \frac{-1}{-1} \\ &= \frac{b}{b-a} \end{aligned}$$

c

$$\begin{aligned} \frac{ax}{b} + c &= 0 \\ \frac{ax}{b} &= -c \\ ax &= -bc \\ x &= -\frac{bc}{a} \end{aligned}$$

d

$$\begin{aligned} px &= qx + 5 \\ px - qx &= 5 \\ x(p-q) &= 5 \\ x &= \frac{5}{p-q} \end{aligned}$$

e

$$\begin{aligned} mx + n &= nx - m \\ mx - nx &= -m - n \\ x(m-n) &= -m - n \\ x &= \frac{-m-n}{m-n} \\ &= \frac{m+n}{n-m} \end{aligned}$$

f

$$\frac{1}{x+a} = \frac{b}{x}$$

Take reciprocals of both sides:

$$\begin{aligned} x + a &= \frac{x}{b} \\ x - \frac{x}{b} &= -a \\ \frac{x}{b} - x &= a \\ \frac{x - xb}{b} &= a \\ \frac{x - xb}{b} \times b &= ab \\ x - xb &= ab \\ x(1-b) &= ab \\ x &= \frac{ab}{1-b} \end{aligned}$$

g $\dfrac{b}{x-a} = \dfrac{2b}{x+a}$

Take reciprocals of both sides:

$$\begin{aligned}\frac{x-a}{b} &= \frac{x+a}{2b}\\ \frac{x-a}{b}\times 2b &= \frac{x+a}{2b}\times 2b\\ 2(x-a) &= x+a\\ 2x-2a &= x+a\\ 2x-x &= a+2a\\ x &= 3a\end{aligned}$$

h

$$\begin{aligned}\frac{x}{m}+n &= \frac{x}{n}+m\\ \frac{x}{m}\times mn + n\times mn &= \frac{x}{n}\times mn + m\times mn\\ nx+mn^2 &= mx+m^2n\\ nx-mx &= m^2n-mn^2\\ x(n-m) &= mn(m-n)\\ x &= \frac{mn(m-n)}{n-m}\end{aligned}$$

Note that $n-m = -m+n$
$\quad = -1(m-n)$

$$\begin{aligned}\therefore \quad x &= \frac{-mn(n-m)}{n-m}\\ &= -mn\end{aligned}$$

i

$$\begin{aligned}-b(ax+b) &= a(bx-a)\\ -abx-b^2 &= abx-a^2\\ -abx-abx &= -a^2+b^2\\ -2abx &= -a^2+b^2\\ x &= -\frac{(-a^2+b^2)}{2ab}\\ &= \frac{a^2-b^2}{2ab}\end{aligned}$$

j

$$\begin{aligned}p^2(1-x)-2pqx &= q^2(1+x)\\ p^2-p^2x-2pqx &= q^2+q^2x\\ -p^2x-2pqx-q^2x &= q^2-p^2\\ -x(p^2+2pq+q^2) &= q^2-p^2\\ x &= \frac{-(q^2-p^2)}{p^2+2pq+q^2}\\ &= \frac{p^2-q^2}{(p+q)^2}\\ &= \frac{(p-q)(p+q)}{(p+q)^2}\\ &= \frac{p-q}{p+q}\end{aligned}$$

k

$$\begin{aligned}\frac{x}{a}-1 &= \frac{x}{b}+2\\ \frac{x}{a}\times ab-ab &= \frac{x}{b}\times ab+2ab\\ bx-ab &= ax+2ab\\ bx-ax &= 2ab+ab\\ x(b-a) &= 3ab\end{aligned}$$

$$x = \frac{3ab}{b-a}$$

l

$$\frac{x}{a-b} + \frac{2x}{a+b} = \frac{1}{a^2-b^2}$$

$$\frac{x(a-b)(a+b)}{a-b} + \frac{2x(a+b)(a-b)}{a+b} = \frac{(a+b)(a-b)}{a^2-b^2}$$

$$x(a+b) + 2x(a-b) = 1$$

$$ax + bx + 2ax - 2bx = 1$$

$$3ax - bx = 1$$

$$x(3a-b) = 1$$

$$x = \frac{1}{3a-b}$$

m

$$\frac{p-qx}{t} + p = \frac{qx-t}{p}$$

$$\frac{pt(p-qx)}{t} + p \times pt = \frac{pt(qx-t)}{p}$$

$$p(p-qx) + p^2t = t(qx-t)$$

$$p^2 - pqx + p^2t = qtx - t^2$$

$$-pqx - qtx = -t^2 - p^2 - p^2t$$

$$-qx(p+t) = -(t^2 + p^2 + p^2t)$$

$$x = \frac{t^2+p^2+p^2t}{q(p+t)} \text{ or } \frac{p^2+p^2t+t^2}{q(p+t)}$$

n

$$\frac{1}{x+a} + \frac{1}{x+2a} = \frac{2}{x+3a}$$

Multiply each term by $(x+a)(x+2a)(x+3a)$.

$$(x+2a)(x+3a) + (x+a)(x+3a) = 2(x+a)(x+2a)$$

$$x^2 + 5ax + 6a^2 + x^2 + 4ax + 3a^2 = 2x^2 + 6ax + 4a^2$$

$$2x^2 + 9ax + 9a^2 = 2x^2 + 6ax + 4a^2$$

$$2x^2 - 9ax - 2x^2 - 6ax = 4a^2 - 9a^2$$

$$3ax = -5a^2$$

$$x = \frac{-5a^2}{3a}$$

$$= -\frac{5a}{3}$$

2 $ax + by = p; bx - ay = q$

Multiply the first equation by a and the second equation by b.

$a^2x + aby = ap$ (1)

$b^2x - aby = bp$ (2)

(1)s + (2):

$$x(a^2+b^2) = ap + bq$$

$$x = \frac{ap+bq}{a^2+b^2}$$

Substitute into $ax + by = p$:

$$a \times \frac{ap + bq}{a^2 + b^2} + by = p$$

$$a(ap + bq) + by(a^2 + b^2) = p(a^2 + b^2)$$

$$a^2p + abq + by(a^2 + b^2) = a^2p + b^2p$$

$$by(a^2 + b^2) = a^2p + b^2p - a^2p - abq$$

$$by(a^2 + b^2) = b^2p - abq$$

$$y = \frac{b(bp - aq)}{b(a^2 + b^2)}$$

$$= \frac{bp - aq}{a^2 + b^2}$$

3 $\frac{x}{a} + \frac{y}{b} = 1; \frac{x}{b} + \frac{y}{a} = 1$

First, multiply both equations by ab, giving the following:

$$bx + ay = ab$$

$$ax + by = ab$$

Multiply the first equation by b and the second equation by a:

$$b^2x + aby = ab^2 \quad (1)$$

$$a^2x + aby = a^2b \quad (2)$$

(1) − (2):

$$x(b^2 - a^2) = ab^2 - a^2b$$

$$x = \frac{ab^2 - a^2b}{b^2 - a^2}$$

$$= \frac{ab(b - a)}{(b - a)(b + a)}$$

$$= \frac{ab}{a + b}$$

Substitute into $bx + ay = ab$:

$$b \times \frac{ab}{a + b} + ay = ab$$

$$\frac{ab^2(a + b)}{a + b} + ay(a + b) = ab(a + b)$$

$$ab^2 + ay(a + b) = a^2b + ab^2$$

$$ay(a + b) = a^2b + ab^2 - ab^2$$

$$ay(a + b) = a^2b$$

$$y = \frac{a^2b}{a(a + b)}$$

$$= \frac{ab}{a + b}$$

4 a Multiply the first equation by b.

$$abx + by = bc \quad (1)$$

$$x + by = d \quad (2)$$

(1) − (2):

$$x(ab - 1) = bc - d$$

$$x = \frac{bc - d}{ab - 1}$$

$$= \frac{d - bc}{1 - ab}$$

It is easier to substitute in the first equation for x:

$$a \times \frac{bc-d}{ab-1} + y = c$$

$$\frac{a(bc-d)(ab-1)}{ab-1} + y(ab-1) = c(ab-1)$$

$$abc - ad + y(ab-1) = abc - c$$

$$y(ab-1) = abc - c$$

$$- abc + ad$$

$$y(ab-1) = -c + ad$$

$$y = \frac{ad-c}{ab-1}$$

$$= \frac{c-ad}{1-ab}$$

b Multiply the first equation by a and the second equation by b.

$a^2x - aby = a^3$ ①

$b^2x - aby = b^3$ ②

① − ②:

$$x(a^2-b^2) = a^3 - b^3$$

$$x = \frac{a^3-b^3}{a^2-b^2}$$

$$= \frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)}$$

$$= \frac{a^2+ab+b^2}{a+b}$$

In this case it is easier to start again, but eliminate x.

Multiply the first equation by b and the second equation by a.

$abx - b^2y = a^2b$ ③

$abx - a^2y = ab^2$ ④

③ − ④:

$$y(-b^2+a^2) = a^2b - ab^2$$

$$y(a^2-b^2) = ab(a-b)$$

$$y = \frac{ab(a-b)}{a^2-b^2}$$

$$= \frac{ab(a-b)}{(a-b)(a+b)}$$

$$= \frac{ab}{a+b}$$

c Add the starting equations:

$$ax + by + ax - by = t + s$$

$$2ax = t + s$$

$$x = \frac{t+s}{2a}$$

Subtract the starting equations:

$$ax + by - (ax - by) = t - s$$

$$2by = t - s$$

$$y = \frac{t-s}{2b}$$

d Multiply the first equation by a and the second equation by b.

$$a^2x + aby = a^3 + 2a^2b - ab^2 \quad ①$$

$$b^2x + aby = a^2b + b^3 \quad ②$$

① − ②:

$$x(a^2 - b^2) = a^3 + a^2b - ab^2 - b^3$$

$$x = \frac{a^3 + a^2b - ab^2 - b^3}{a^2 - b^2}$$

$$= \frac{a^2(a+b) - b^2(a+b)}{a^2 - b^2}$$

$$= \frac{(a^2 - b^2)(a+b)}{a^2 - b^2}$$

$$= a + b$$

Substitute into the second, simpler equation.

$$b(a+b) + ay = a^2 + b^2$$

$$ab + b^2 + ay = a^2 + b^2$$

$$ay = a^2 + b^2 - ab - b^2$$

$$ay = a^2 - ab$$

$$y = \frac{a^2 - ab}{a}$$

$$= a - b$$

e Rewrite the second equation, then multiply the first equation by $b + c$ and the second equation by c.

$$(a+b)(b+c)x + c(c+c)y = bc(b+c) \quad ①$$

$$acx + c(b+c)y = -abc \quad ②$$

① − ②:

$$x((a+b)(b+c) - ac) = bc(b+c) + abc$$

$$x(ab + ac + b^2 + bc - ac) = bc(b+c+a)$$

$$x(ab + b^2 + bc) = bc(a+b+c)$$

$$xb(a+b+c) = bc(a+b+c)$$

$$x = \frac{bc(a+b+c)}{b(a+b+c)}$$

$$= c$$

Substitute into the first equation. (It has the simpler y term.)

$$c(a+b) + cy = bc$$

$$ac + bc + cy = bc$$

$$cy = bc - ac - bc$$

$$cy = -ac$$

$$y = \frac{-ac}{c}$$

$$= -a$$

f First simplify the equations.

$$3x - 3a - 2y - 2a = 5 - 4a$$

$$3x - 2y = 5 - 4a + 3a + 2a$$

$$3x - 2y = a + 5 \quad ①$$

$$2x + 2a + 3y - 3a = 4a - 1$$

$$2x + 3y = 4a - 1 - 2a + 3a$$

$$2x + 3y = 5a - 1 \quad ②$$

Multiply ① by 3 and ② by 2.

$9x - 6y = 3a + 15 \quad (3)$

$4x + 6y = 10a - 2 \quad (4)$

$(3) + (4)$:

$$13x = 13a + 13$$
$$x = a + 1$$

Substitute into (2):

$$\begin{aligned} 2(a+1) + 3y &= 5a - 1 \\ 2a + 2 + 3y &= 5a - 1 \\ 3y &= 5a - 1 - 2a - 2 \\ 3y &= 3a - 3 \\ y &= a - 1 \end{aligned}$$

5 a
$$\begin{aligned} s &= ah \\ &= a(2a+1) \end{aligned}$$

b Make h the subject of the second equation.

$$\begin{aligned} h &= a(2+h) \\ &= 2a + ah \\ h - ah &= 2a \\ h(1-a) &= 2a \\ h &= \frac{2a}{1-a} \end{aligned}$$

Substitute into the first equation.

$$\begin{aligned} s &= ah \\ &= a \times \frac{2a}{1-a} \\ &= \frac{2a^2}{1-a} \end{aligned}$$

c
$$\begin{aligned} h + ah &= 1 \\ h(1+a) &= 1 \\ h &= \frac{1}{(1+a)} = \frac{1}{a+1} \\ as &= a + h \\ &= a + \frac{1}{a+1} \\ &= \frac{a(a+1)+1}{a+1} \\ &= \frac{a^2+a+1}{a+1} \\ s &= \frac{a^2+a+1}{a(a+1)} \end{aligned}$$

d Make h the subject of the second equation.

$$\begin{aligned} ah &= a + h \\ ah - h &= a \\ h(a-1) &= a \\ h &= \frac{1}{a-1} \end{aligned}$$

Substitute into the first equation.

$$\begin{aligned} as &= s + h \\ as &= s + \frac{a}{a-1} \\ as - s &= \frac{a}{a-1} \\ s(a-1) &= \frac{a}{a-1} \\ s(a-1)(a-1) &= \frac{a(a-1)}{a-1} \\ s(a-1)^2 &= a \\ s &= \frac{a}{(a-1)^2} \end{aligned}$$

e $$\begin{aligned} s &= h^2 + ah \\ &= (3a^2)^2 + a(3a^2) \\ &= 9a^4 + 3a^3 \\ &= 3a^3(3a+1) \end{aligned}$$

f $$\begin{aligned} as &= a + 2h \\ &= a + 2(a-s) \\ &= a + 2a - 2s \\ as + 2s &= 3a \\ s(a+2) &= 3a \\ s &= \frac{3a}{a+2} \end{aligned}$$

g $$\begin{aligned} s &= 2 + ah + h^2 \\ &= 2 + a\left(a - \frac{1}{a}\right) + \left(a - \frac{1}{a}\right)^2 \\ &= 2 + a^2 - 1 + a^2 - 2 + \frac{1}{a^2} \\ &= 2a^2 - 1 + \frac{1}{a^2} \end{aligned}$$

h Make h the subject of the second equation.

$$\begin{aligned} as + 2h &= 3a \\ 2h &= 3a - as \\ h &= \frac{3a - as}{2} \end{aligned}$$

Substitute into the first equation.

$$\begin{aligned} 3s - ah &= a^2 \\ 3s - \frac{a(3a - as)}{2} &= a^2 \\ 6s - a(3a - as) &= 2a^2 \\ 6s - 3a^2 + a^2 s &= 2a^2 \\ a^2 s + 6s &= 2a^2 + 3a^2 \\ s(a^2 + 6) &= 5a^2 \\ s &= \frac{5a^2}{a^2 + 6} \end{aligned}$$